A METHOD FOR SOLVING THE MINIMIZATION OF THE MAXIMUM NUMBER OF OPEN STACKS PROBLEM WITHIN A CUTTING PROCESS

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ABSTRACT

In this paper, the problem of minimizing the maximum number of open stacks around a saw machine is addressed. A new heuristic and a branch-and-bound based exact method for the problem are presented.

DEDICATION

This thesis is dedicated to my parents.

ACKNOWLEDGMENTS

I would like to thank Dr. John Karlof, my advisor, for his constant support. I am also thankful to Dr. Matthew TenHuisen and Dr. Jeffrey L. Brown, my committee members, for their help with this thesis.

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1 INTRODUCTION

1.1 Scope and purpose

In an industrial environment, one general problem that occurs is how to find a particular sequence of production jobs that minimizes production costs. The mathematical solutions of this type of problem have been the interest of various researchers. The problem that this paper addresses is to determine a method that minimizes the maximum number of open orders of clients. This problem is known as the minimization of open stack problem or MOSP.

The industrial problem that is considered here arises in a production setting which consists of a set of products and a set of customer's orders. For instance, in the glass industry where different piece types (set of products) are cut that are used for car or office windows. Suppose that the cutting patterns (set of customer's orders) have already been determined around a saw machine. Each pattern is composed of piece types and as the patterns are cut, the pieces of the same type are stacked together. However, space around the vicinity is limited and hence, the number of distinct stacks should be small. It is assumed that a stack is open as soon as a new piece type is cut and it remains open until all the piece types corresponding to that stack are cut, only then can the stack be removed from the vicinity. This rule has obvious implications on handling costs and it also minimizes risks. Thus, it is necessary to minimize the maximum number of open stacks during the whole production run.

1.2 Previous work

Dyson and Gregory [2] discuss the cut pattern sequence problem aiming to reduce the number of discontinuities among piece types to be cut, i.e. if any piece type does not occur in the next pattern in the sequence, then a cutting discontinuity exists.

Madsen [3,4] presents a procedure to reduce the cut distance among equal piece types and an objective to minimize the distance between corresponding piece types, called the order spread. A spread is the number of different processed patterns between the first and the last piece types.

Lins [5] introduces a limited version of the MOSP where a pattern contains two piece types. Fink and Voβ [6] introduce a heuristic approach to solve the MOSP and the order spread minimization. Foerster and W$\ddot{a}$scher [7] analyze three heuristics in solving for order spread minimization problem in sequencing cutting patterns.

Yuen [8,9] presents six heuristics for solving the MOSP where the third heuristic, the Yuen3, has a very good practical behavior. Yuen and Richardson [11] present the full explanation and some optimality conditions for these six heuristics. Faggioli and Bentivoglio [10] yield a two-phase approach in solving the MOSP, the implementation of the heuristic procedure and the enumeration on the patterns using a branch and bound approach.

Yanasse [1,12] and Limeira [13] present the major concepts in the formulation for the method presented in this paper. Faggioli and Bentivoglio [14] present the formulation of MOSP as a linear integer programming (LIP), but Becceneri [15] refers that a direct attack through LIP may be infeasible which is reinforced by Yannase [16] that the problem is proven to be NP-Hard.

1.3 Open stacks problem

Consider a production setting shown in table 1, the piece types contained in each pattern is presented. The number of times that a piece type is cut in each pattern is not indicated since it is not relevant for the MOSP. For pattern P_1, observe that three stacks will be opened, one for p_1, one for p_2 and another for p_4. The same number of stacks would be open if there is a pattern with two piece types of p_1, five piece types of p_2 or three piece types of p_4. Observe that when closing a stack all

Table 1: Set of cutting process

Pattern	Piece type
P_1	$p_1p_2p_4$
P_2	$p_2p_4p_5p_6$
P_3	p_3p_4
P_4	$p_1p_3p_5$
P_5	$p_1p_4p_5$
P_6	p_5p_6

Table 2: Stacks observed following sequence

Pattern sequenced	Open piece types	Closed piece types	Number of open stacks
P_1	$p_1p_2p_4$		3
P_2	$p_1p_2p_4p_5p_6$	Finished p_2	5
P_3	$p_1p_3p_4p_5p_6$		5
P_4	$p_1p_3p_4p_5p_6$	Finished p_3	5
P_5	$p_1p_4p_5p_6$	Finished p_1 and p_4	4
P_6	p_5p_6	Finished p_5 and p_6	2

patterns containing the corresponding piece must have been cut.

Suppose that we want to find the maximum number of open stacks for sequence $P_1P_2P_3P_4P_5P_6$. We start cutting piece types $p_1p_2p_4$ of pattern P_1 with three open stacks. Pieces of pattern P_2 are to be cut next where the stack of piece type p_2 is already finished at this stage with five open stacks. Following this idea, we end up with a maximum number of open stacks of five as illustrated in table 2.

Another possible sequence is $P_3P_4P_5P_1P_2P_6$ with a maximum of four open stacks. Observe that for this 6-pattern problem, there are already $6! = 720$ possible different sequences to check which one minimizes the maximum number of open stacks. Thus, we are interested in finding the optimal sequence that yields the minimum of the maximum number of open stacks.

Table 3: Data presented in table 1 into a matrix

$$P_{i,j} = \left[\begin{array}{c|cccccc} & p_1 & p_2 & p_3 & p_4 & p_5 & p_6 \\ \hline P_1 & 1 & 1 & 0 & 1 & 0 & 0 \\ P_2 & 0 & 1 & 0 & 1 & 1 & 1 \\ P_3 & 0 & 0 & 1 & 1 & 0 & 0 \\ P_4 & 1 & 0 & 1 & 0 & 1 & 0 \\ P_5 & 1 & 0 & 0 & 1 & 1 & 0 \\ P_6 & 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

1.4 Binary matrix presentation

In the production setting with I distinct patterns given, each one of these patterns contains a combination of at most J piece types. The input of the data can be given by an I x J binary matrix P, representing patterns as rows and piece types as columns where

$$P_{i,j} = \begin{cases} 1 & \text{if piece type } j \text{ is to be cut from the pattern } i, \\ 0 & \text{otherwise.} \end{cases}$$

The data in table 1 as a binary matrix is illustrated in table 3. The first entry 1 means that in pattern P_1, piece type p_1 is present. Thus, row 1 as pattern P_1 contains piece types $p_1p_2p_4$.

1.5 Graph model

The input data for the MOSP can be given by a graph. In graph $G(V, E)$, a node $k \in V$ represents the piece type p_k and two nodes i and j are adjacent (that is, $(i, j) \in E$) if and only if the corresponding piece types p_i and p_j are simultaneously present in the same pattern.

The alteration of the patterns given in table 1 into a graph is illustrated in figure 1. The nodes 1, 2, 3, 4, 5, 6 represent the piece types $p_1, p_2, p_3, p_4, p_5, p_6$, respectively.

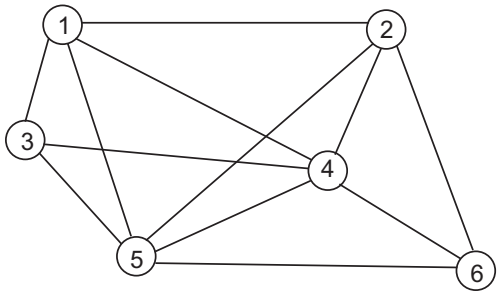


Figure 1: Graph of Table 1

Pattern P_1 contains arcs (1,2), (1,4), and (2,4), P_2 contains arcs (2,4), (2,5), (2,6), (4,5), (4,6), (5,6), and so on.

1.6 Outline of the paper

In this paper, we consider an MOSP with at most two piece types a pattern. The method for solving the MOSP is the incorporation of the algorithms presented in the next sections. Each algorithm is illustrated with graphical examples.

The next sections are presented as follows. In section 2, we have the pre-processing procedures on the input data that simplify the problem, and the algorithms for trees, simple cycles and 1-trees are presented. In section 3, we present an algorithm for the heuristic minimal cost node of a minor of graph G that uses the idea of visiting all arcs in G_p, a minor of G. In section 4, we present the exact method in a branch-and-bound scheme. This section is subdivided into 4 subsections with the inclusion of the detailed implementation of a lower bound. Finally, in section 5, we present the conclusion and future work.

2 Pre-processing phase

In this section, we present the two main objectives of the pre-processing that use the idea of a clique in a graph. We begin with a definition and then the propositions and algorithms in the following subsections.

Definition 1 *A clique in a graph G is a complete subgraph K_h of G. The size of a clique is the number of vertices it contains which is h.*

It is assumed, without loss of generality, that the input data for MOSP is given by a matrix or a graph as illustrated in subsections 1.4 and 1.5.

A pattern P_i with h different piece types defines a clique K_h of size h and it represents an obvious optimal solution when the piece types of the rest of the patterns in MOSP are contained in P_i. In addition, an optimal solution to the problem can also be examined as a succession of large cliques.

The two main objectives in this phase are (i) to eliminate redundancies among the patterns and (ii) to detect conditions for which a solution can be found in polynomial time.

(i) A redundancy occurs whenever in G, a clique is a sub-graph of another larger one. For instance, in table 1, $P_6 \subseteq P_2$, this means that P_6 can be removed during an optimal solution search of the problem, since P_6 can be inserted immediately after P_2 in an optimal sequence and no more stacks will be open. We say, P_2 dominates P_6. The dominance of a pattern happens whenever $P_j \subseteq P_i$, where $i \neq j$.

(ii) After eliminating the redundancies among the patterns, we check whether the sub-graphs of G have trees, simple cycles or 1-trees as components. If such components occur, we apply the algorithms presented in the next subsection to generate an optimal sequence for the piece types contained in these components. The main idea is to explore the structure of graphs which possess "small" cliques in order to find its lower bound that can be the basis for the optimal solution. Generally, an optimal solution to MOSP is given by a sequence of possibly large and overlapping cliques, connected by components such as trees. Hence, it is important to exactly solve the schedule of the latter components in polynomial time to incorporate them into the final solution.

Table 4: Reverse sequence of $P_1P_2P_3P_4P_5P_6$

Pattern sequenced	Open piece types	Closed piece types	Number of open stacks
P_6	p_5p_6		2
P_5	$p_1p_4p_5p_6$		4
P_4	$p_1p_3p_4p_5p_6$		5
P_3	$p_1p_3p_4p_5p_6$	Finished p_3	5
P_2	$p_1p_2p_4p_5p_6$	Finished p_5 and p_6	5
P_1	$p_1p_2p_4$	Finished p_1, p_2 and p_4	3

2.1 Propositions

The algorithms presented in the next subsection are based on the following propositions considering that a pattern contains at most two piece types. Each proposition is illustrated by a simple example.

Proposition 1 *Let L be the list of open piece types with a possible sequencing of the patterns of a MOSP, $P_1P_2 \dots P_n$. The reverse order sequence of the patterns $P_n \dots P_2P_1$ produces the same open piece types in reverse order, that is, list L in reverse order.*

Proof. Suppose the pattern sequence $P_1P_2 \dots P_n$ generates a list of open piece types L $= l_1l_2 \dots l_n$ where l_i represents the list of open piece types in P_i. Then the reverse pattern sequence $P_n \dots P_2P_1$ generates a list of open piece types $l_n \dots l_2l_1$, that is, list L in reverse order. □

To illustrate the proposition, consider for instance table 2 with a pattern sequence $P_1P_2P_3P_4P_5P_6$. Column 2 of table 2 lists the open piece types which generates a maximum number of open stack 5. Table 4 shows that the reverse order of the patterns lists the reverse order of the open piece types (see column 2)with the same maximum number of open stacks.

Proposition 2 *Suppose we have a MOSP, $PROB_1$, with N patterns to be sequenced. Suppose another problem, $PROB_2$, with the same N patterns as problem $PROB_1$ plus a few more patterns. Then the optimal solution value of $PROB_2$ is greater than or equal to the optimal solution value of $PROB_1$.*

Proof. Trivial. Take the optimal solution of $PROB_2$ and obtain a feasible solution to $PROB_1$ by simply ignoring, in the sequence, the patterns that belong only to $PROB_2$. □

In table 1, pattern P_6 is dominated by P_2. Removing pattern P_6 from the problem still yields an optimal solution of 5. This illustrates proposition 2.

Proposition 3 *Let $G(V, E)$, the graph obtained from a problem P, be a connected non-empty graph where the nodes have minimum degree $n (n \geq 2)$. Then, a lower bound on the optimal solution value to P is $n + 1$, that is, in an optimal solution to P, we must have at least $n + 1$ open stacks at some time during cutting.*

Proof. For a node of degree n, its piece type will be contained in n patterns since all piece types are contained in at most two patterns. Then, for any sequencing of the patterns, a piece type, say i, will remain open until all n or more patterns containing it are cut, that is, all arcs incident to node i are sequenced. Each one of these arcs is incident to a different node, say j, corresponding to another piece type, which in turn, remains unfinished unless all n or more patterns containing it are sequenced. All arcs incident to any node will have to be sequenced sometime. Since no node can be finished unless all its n (or more) incident arcs are sequenced and since each one of them creates an open stack, the proposition holds. □

Take for instance, the graph of figure 1. Node 6 has a minimum degree 3 and the rest of the nodes have degree greater than or equal to 3. Therefore, any feasible sequencing of the patterns will produce at least 4 open stacks at some time during the cutting.

2.2 Algorithms for trees, simple cycles, and 1-trees as components

In this subsection, we present an algorithm for a few special cases of MOSP. We begin with the simplest case and proceed to more complicated ones. Definitions and procedures are introduced for easy understanding.

Case 1 *Graph with a single node.*

This case corresponds to having a pattern with one piece type. This is trivial.

Case 2 *Special tree I - A tree with all nodes having degree 2 or less*

If the graph has a single node then we are in case 1. Else, we sequence the patterns in the following procedure.

Procedure WALK

Start with a pattern that finishes a piece type completely, that is, an arc incident to a single node with degree 1 and keep sequencing next the pattern that ends any open piece type.

Definition 2 *The sequence of patterns obtained using procedure WALK is called walk W. A walk revW is the reverse order sequence of patterns of W. The maximum number of open piece types of a walk W is its degree denoted by w.*

Consider the graph in figure 2, we start with pattern P_1 since node 1 has degree 1, then patterns P_2P_3 or we start with pattern P_3, then patterns P_2P_1. The procedure WALK applied yields a walk $W = P_1P_2P_3 = P_3P_2P_1$ with $w = 2$. Either of the sequencing results in an optimal solution as explained in proposition 1.

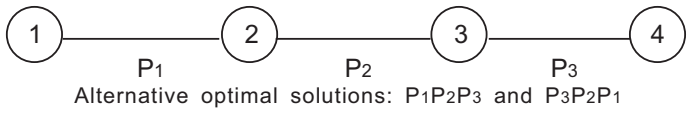
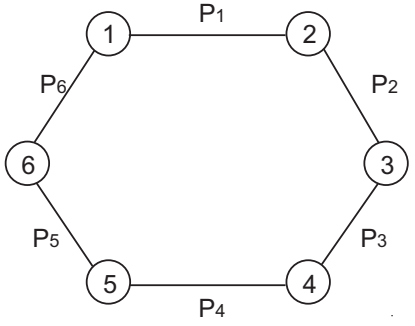


Figure 2: Illustration of procedure WALK in special tree 1

Case 3 *Simple Cycle (a polygon) - nonempty graph with all nodes of degree 2*

In this case, we follow the strategy of sequencing patterns in order to finish any open piece types. Any pattern can be chosen to start with the sequencing.

For instance in figure 3, the easiest way to find an optimal solution is to follow the sequence $P_1P_2P_3P_4P_5P_6$ or its reverse order with a maximum of 3 open stacks according to proposition 3. We can also have the same optimal solution for sequence $P_1P_6P_2P_3P_4P_5$.



Alternative optimal solutions: P1P2P3P4P5P6 or P6P5P4P3P2P1 or P1P6P5P4P3P2

Figure 3: Illustration of a simple cycle

Case 4 *Special tree II - A tree with all nodes of degree two or less with the exception of at most one of degree 3.*

For this case, we introduce a another procedure that is also needed in the next cases.

Procedure CONSTRUCT

Let Γ_k be the set of walks with the same end node k to be sequenced. If $|\Gamma_k| \geq 2$ then deg(k) ≥ 3. The sequence of walks must begin with walk W_i and end with walk $revW_j$ where $W_i, W_j \in \Gamma_k$ and the degrees of W_i and W_j are respectively the largest and the second largest.

Algorithm tree II

Apply procedure WALK starting with a node of degree 1 and end with a node of degree greater than 2. When the end node of walk W has degree greater than 2 then degree w excludes the end node. With the walks obtained, apply procedure CONSTRUCT.

Consider the graph in figure 4, we can obtain three walks with the same end node 5. Suppose we have $W_1 = P_1P_2$ with $w_1 = 2$, $W_2 = P_6$ with $w_2 = 1$, and $W_3 = P_5P_4P_3$ with $w_3 = 2$. Then applying procedure CONSTRUCT, we have an optimal sequence of walks $W_1W_2\text{rev}W_3$ or $W_3W_2\text{rev}W_1$ equivalent to the sequence of patterns $P_1P_2P_6P_3P_4P_5$ or $P_5P_4P_3P_6P_2P_1$, respectively. The maximum number of open stacks is 2.

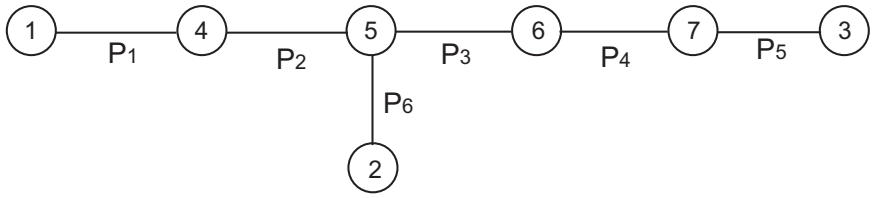


Figure 4: Illustration of special tree II

Case 5 *Trees*

The algorithm for trees uses a similar concept as the algorithm tree II. We construct a solution for the case trees by adding together the optimal solutions of its subtrees. We start with a definition of a score that is needed throughout the algorithm. Procedures WALK and CONSTRUCT are also applied.

Definition 3 *Let Γ_k be a set of walks with the same end node k. The score of k is $s(k) = max\{w\text{: } W \in \Gamma_k\} + ADJUSTMENT$ where*

$$ADJUSTMENT = \begin{cases} 1 & \textit{if more than one walk have the same maximum} \\ & \textit{degree(w)}, \\ 0 & \textit{otherwise.} \end{cases}$$

Let W_i denote the walk i corresponding to $subtree_i$ and w_i denote its corresponding degree. Suppose that in figure 5, only walks W_1, W_2 and W_3 are considered so far with end node k. If for instance, we have $w_1 = w_3 = 2$ and $w_2 = 1$, then the score of k is three. In this instance, the score of k provides an approximate minimum value of the maximum number of open stacks that will result in any optimal sequencing of the patterns in the subtrees having end node k as the root node.

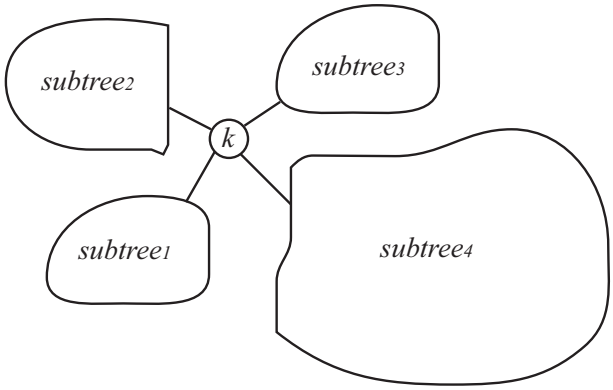


Figure 5: Computing the score of k

Algorithm TREES

For this case, we follow four steps. We denote L as a set of nodes with degree 1.

Step 0 (Initialization) Set the score of all nodes in the graph equal to 0. List (L) all nodes with degree 1.

Step 1 Using the procedure WALK, determine the walks W_i, $i = 1, 2, \ldots, n$, starting from each node in L and end with a node of degree greater than two. Include its corresponding degree w_i and end node k_i. If a walk contains only a pattern then its degree $w = 1$.

Step 2 Delete all the nodes and patterns used in W_i from the graph excluding their end nodes. With the resulting graph, list (L) all nodes with degree 1 and compute the score of each end node k_i from the walks obtained.

Step 3 If there is only one node k in L, construct a final sequence of walks with end node k, following procedure CONSTRUCT and STOP. Otherwise, among the nodes in the list choose one, say k, that has the smallest score. Break ties at random. Then

3.1 From the resulting graph obtained in step 2, create a walk starting from node k and end with a node of degree greater than two. If node k is adjacent to a node with zero score, then this walk must be affixed right after the sequence of walks constructed in 3.2. Otherwise, affixing is not needed.

3.2 Using procedure CONSTRUCT, create a walk equal to a sequence of patterns using the walks obtained having an end node k. Affix to this sequence the pattern or patterns equated from the walk created in 3.1 and consider this as an "extended" walk. Calculate the degree w associated to this "extended" walk. Store this extended walk with its corresponding degree w and end node.

3.3 Delete all the walks used in obtaining the extended walk then return to step 2.

Suppose we have to sequence 22 patterns with two piece types per pattern as presented in table 5. The graph of this table is in figure 6.

The following are the steps of the algorithm that are applied in figure 6.

Step 0: Set each node score equal to 0. L = {1, 4, 10, 12, 14, 18, 19, 21, 23}.

Step 1: Walks determined from each node in L with degree 1.

Table 5: Set of patterns to be sequenced

Pattern	Piece Type	Pattern	Piece Type
P_1	p_1p_2	P_{12}	p_6p_{13}
P_2	p_2p_3	P_{13}	$p_{13}p_{14}$
P_3	p_3p_4	P_{15}	$p_{15}p_{16}$
P_4	p_3p_5	P_{16}	$p_{16}p_{17}$
P_5	p_5p_6	P_{17}	$p_{17}p_{18}$
P_6	p_6p_7	P_{18}	$p_{16}p_{19}$
P_7	p_7p_8	P_{19}	$p_{16}p_{20}$
P_8	p_8p_9	P_{20}	$p_{20}p_{21}$
P_9	p_9p_{10}	P_{21}	$p_{15}p_{22}$
P_{10}	p_8p_{11}	P_{22}	$p_{22}p_{23}$
P_{11}	$p_{11}p_{12}$		

starting node	W_i	w_i	end node
1	$W_1 = P_1P_2$	$w_1 = 2$	3
4	$W_2 = P_3$	$w_2 = 1$	3
10	$W_3 = P_9P_8$	$w_3 = 2$	8
12	$W_4 = P_{11}P_{10}$	$w_4 = 2$	8
14	$W_5 = P_{13}P_{12}$	$w_5 = 2$	6
18	$W_6 = P_{17}P_{16}$	$w_6 = 2$	16
19	$W_7 = P_{18}$	$w_7 = 1$	16
21	$W_8 = P_{20}P_{19}$	$w_8 = 2$	16
23	$W_9 = P_{22}P_{21}$	$w_9 = 2$	15

Step 2: Delete all the nodes and patterns used from W_1 to W_9 excluding their end nodes. The resulting graph is in figure 7. L = {8, 16}.

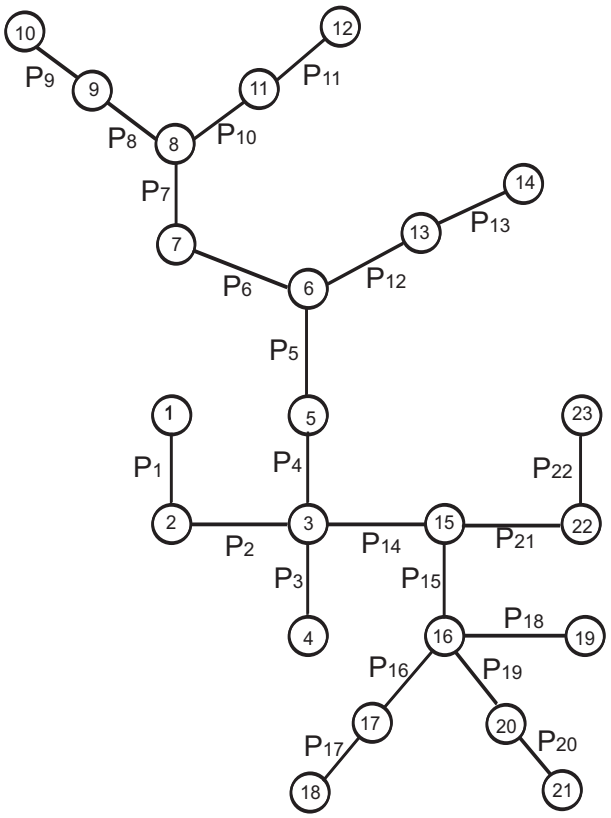


Figure 6: Tree corresponding to data in Table 5

End node	Score
3	2
6	2
8	3
15	2
16	3

Step 3: End nodes 8 and 16 have the same score. We select $k = 8$.

3.1 Starting from node 8, $W_{10} = P_7P_6$ with $w_{10} = 2$ and end node 6.

Since s(7) = 0, the score of the adjacent node, affixing is needed.

3.2 Using procedure CONSTRUCT: $W_{11} = W_3\text{rev}W_4W_{10} = P_9P_8P_{10}P_{11}P_7P_6$ with $w_{11} = 3$ and end node 6.

3.3 Walks deleted: $W_3W_4W_{10}$. Return to step 2.

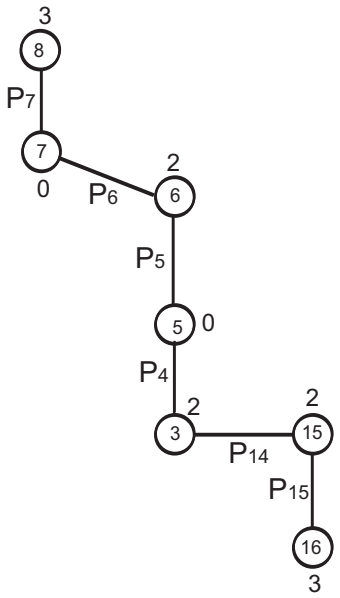


Figure 7: Graph after deleting nodes and patterns from W_1 to W_9

Step 2: Delete nodes 7, 8 and patterns P_6P_7, the resulting graph is in figure 8. L = {6, 16}.

End node	Score
3	2
6	3
15	2
16	3

Step 3: $k = 6$

3.1 Starting from node 6, $W_{12} = P_5P_4$ with $w_{12} = 2$ and end node 3.

Since s(5) = 0, then affixing is needed.

3.2 Using procedure CONSTRUCT: $W_{13} = W_{11}\text{rev}W_5W_{12} = P_9P_8P_{10}P_{11}P_7P_6P_{12}P_{13}P_5P_4$ with $w_{13} = 3$ and end node 3.

3.3 Walks deleted: $W_5W_{11}W_{12}$. Return to step 2.

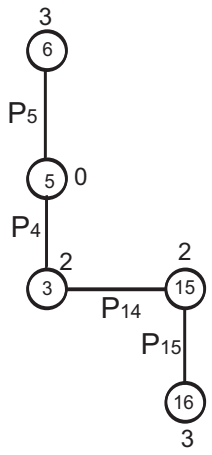


Figure 8: Graph after deleting node 8 and patterns P_6P_7

Step 2: Delete nodes 5, 6 and patterns P_4P_5, the resulting graph is in figure 9. L = {3, 16}.

End node	Score
3	3
15	2
16	3

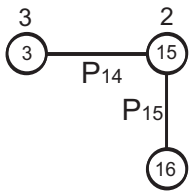


Figure 9: Graph after deleting node 6 and patterns P_4P_5

Step 3: $k = 3$

3.1 Starting from node 3, $W_{14} = P_{14}$ with $w_{14} = 1$ and end node 15.

Adjacent node 15 has score equal to 2. Affixing is not needed.

3.2 Using procedure CONSTRUCT: $W_{15} = P_9P_8P_{10}P_{11}P_7P_6P_{12}P_{13}P_5P_4P_3P_2P_1$ with

$w_{15} = 3$ and end node 15.

3.3 Walks deleted: $W_1W_2W_{13}$. Return to step 2.

Step 2: Delete node 3 and pattern P_{14}, the resulting graph is left with nodes 15 and 16 connected by pattern P_{15}. L = {15, 16}.

End node	Score
15	2
16	3

Step 3: $k = 15$

3.1 Starting from node 15, $W_{16} = P_{15}$ with $w_{16} = 1$ and end node 16.

Adjacent node 3 has score equal to 3. Affixing is not needed.

3.2 Using procedure CONSTRUCT: $W_{17} = P_9P_8P_{10}P_{11}P_7P_6P_{12}P_{13}P_5P_4P_3P_2P_1P_{14}P_{21}P_{22}$ with $w_{17} = 3$ and end node 15.

3.3 Walks deleted: $W_9W_{14}W_{15}$. Return to step 2.

Step 2: Delete node 15 and pattern P_{15}, the resulting graph is left with a single node 16. L = {16}.

Step 3: $k = 16$, using procedure CONSTRUCT then we have one possible optimal sequence equal to $W_{17}W_6W_7W_{16}\text{rev}W_8 = P_9P_8P_{10}P_{11}P_7P_6P_{12}P_{13}P_5P_4P_3P_2P_1P_{14}P_{21}P_{22}$ $P_{17}P_{16}P_{18}P_{15}P_{19}P_{20}$ with a maximum number of open stacks equal to 3.

Case 6 *1-Trees - "Almost trees"*

Definition 4 *A 1-tree is formed by connecting a tree with a node outside the tree with two arcs so that a cycle is formed.*

For this case, we follow the same steps in algorithm TREES until reaching a polygon subgraph. Then we apply a new procedure, procedure POLYGON. Procedure POLYGON has three modified steps from algorithm TREES and two other procedures, procedure CONSTRUCT1 and procedure CONSTRUCT2. The two other procedures are constructed to ensure that the maximum number of open stacks is kept to the least possible value.

Algorithm 1-TREES

We suppose that the resulting graph is a polygon after using the algorithm for TREES. We denote L_i as a set of walks where $i = 1, 2$, and 3.

Procedure POLYGON

Select a node k_1 with the largest score in the polygon. Split the list of walks obtained from using algorithm TREES into two lists, L_1 and L_2, in descending order of degree w where L_1 contains walks with end node k_1 and L_2 contains the remaining walks. Consequently, we have one of the following cases with its procedure in generating an optimal sequencing:

(i) L_1 and L_2 are empty.

Apply the procedure for the cycle case.

(ii) L_1 is non-empty but L_2 is empty.

Starting from a pattern in the polygon incident to node k_1, determine a sequence of patterns using the procedure in cycle case and consider this as a walk with degree $w = 2$ and end node k_1. Then apply procedure CONSTRUCT using all the walks obtained with end node k_1.

(iii) L_1 and L_2 are non-empty.

Let P_a and P_b be the two incident patterns to node k_1 in the polygon. Introduce an artificial node k_a incident to P_a, separating node k_1 and its incident pattern

P_b so that the resulting graph forms a tree. Set the score of k_1 and k_a equal to zero. Then apply the following modified steps:

Step 1 Using procedure WALK, determine new walks W_i and W_{i+1} starting from nodes k_1 and k_a with end nodes that have scores greater than zero. These walks W_i and W_{i+1} are the succeeding walks from the walks obtained using algorithm TREES. Include to these new walks their corresponding degrees w and their end nodes. List these new walks as L_3. If a walk contains only a pattern then its degree $w = 1$.

Step 2 Delete all the nodes and patterns used from the walks newly obtained in the graph excluding their end nodes. With the resulting graph, list (L) all nodes with degree 1 and compute the score of each end node from all the walks obtained.

Step 3 If there is only one node k in L, construct a final sequence of walks with end node k using all the walks listed in L_1, L_2, and L_3, following procedure CONSTRUCT1 and STOP. Otherwise, among the nodes in L choose one, say k, that has the smallest score. Then

3.1 From the resulting graph obtained in step 2, create a walk starting from node k and end with a node's score greater than zero. Add this walk to L_2. If node k is adjacent to a node with zero score, then this walk must be affixed right after the sequence of walks constructed in 3.2. Otherwise, affixing is not needed.

3.2 Using procedure CONSTRUCT2, create a walk equal to a sequence of patterns using the walks obtained having an end node k. Affix to this sequence the pattern or patterns equated from the walk created in 3.1 and consider this as an "extended" walk. Calculate the degree w associated to this "extended"

walk. Store this extended walk with its corresponding degree w and end node. Add this walk to L_3.

3.3 Delete all the walks used in obtaining the "extended" walk, rewrite L_2 and L_3 with the new walks excluding the deleted walks then return to step 2.

Procedure CONSTRUCT1

Let W_v be the walk having the largest degree w in L_2. Let W_s and W_t be the last walks listed in L_3 with end node k where the degree $w_t \geq w_s$. Let $L_1 = \{W_c, W_f, \ldots, W_e\}$ where $W_c, W_f, \ldots, W_e$ are in descending order of degree w. If degree $w_c = w_f$ and it is greater than w_t and w_v, i.e. $w_c = w_f > w_t$ and $w_c = w_f > w_v$, then

a. Construct a sequence of walks with end node k_1 from L_1 using procedure CONSTRUCT;

b. Construct a sequence of walks with end node k from L_2 and L_3 using procedure CONSTRUCT where the second walk in the sequence must have the first pattern that is connected to k_1;

c. Construct a final sequence by affixing the sequence in **b** to **a**.

Otherwise, start with a sequence from $W_c, W_f, \ldots, W_e W_t W_s$, then all walks in L_2 and end with revW_v.

Procedure CONSTRUCT2

Let k be the node selected. Let W_s and W_t be the walks with end node k having the largest degree w from L_2 and L_3, respectively. Break ties at random. If $w_t \geq w_s$ then a sequence of walks is $W_t W_i W_l \ldots W_j \text{rev}W_s$ where $W_i W_l \ldots W_j$ are the walks in descending order of degree w from L_2 with end node k. Otherwise, $W_s W_i W_l \ldots W_j \text{rev}W_t$.

The following are examples in finding the optimal sequence of a 1-tree graph:

Example 1. L_1 is non-empty but L_2 is empty.

Given a graph in figure 10, we begin by using the algorithm TREES.

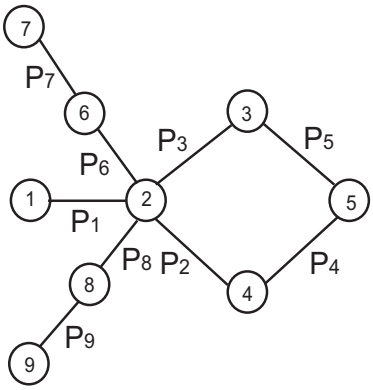


Figure 10: 1-tree where L_2 is empty when procedure POLYGON is applied

Step 0: Each node score is set to 0. L = {1, 7, 9}.

Step 1: Walks determined from each node in L with degree 1.

starting node	W_i	w_i	end node
1	$W_1 = P_1$	$w_1 = 1$	2
7	$W_2 = P_7P_6$	$w_2 = 2$	2
9	$W_3 = P_9P_8$	$w_3 = 2$	2

Step 2: Delete all the nodes and patterns used from W_1 to W_3 excluding their end nodes. The resulting graph is a polygon. L = ϕ.

End node	Score
2	3

Procedure POLYGON: $k_1 = 2$. $L_1 = \{W_3, W_2, W_1\}$ and $L_2 = \phi$.

Starting from node 2, $W_4 = P_3P_5P_4P_2$ with $w_4 = 2$ and end node 2.

Procedure CONSTRUCT: $W_2W_3W_1\text{rev}W_4 = P_7P_6P_9P_8P_1P_2P_4P_5P_3$ with a maximum number of open stacks equal to 3.

Example 2. L_1 and L_2 are non-empty.

Consider the graph in figure 11.

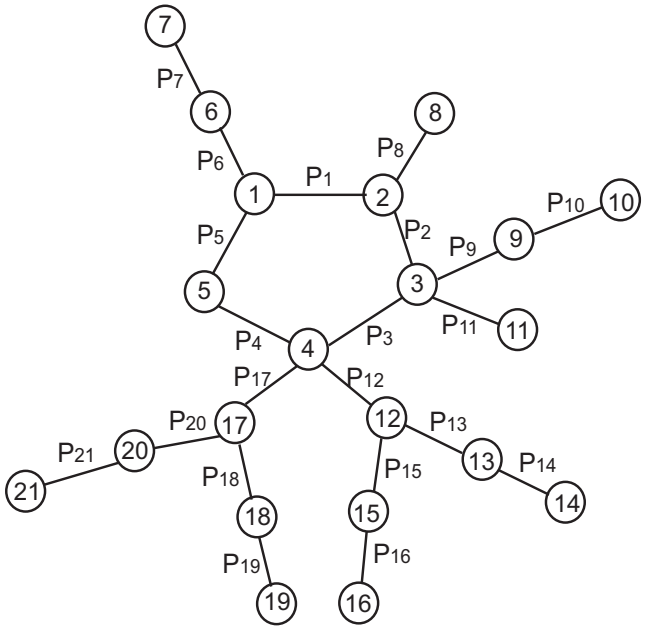


Figure 11: 1-tree where L_2 is non-empty when procedure POLYGON is applied

After using algorithm TREES, the resulting graph is a polygon. The following are the walks left and their end node's scores:

From step 1 using the algorithm TREES:

starting node	W_i	w_i	end node
7	$W_1 = P_7P_6$	$w_1 = 2$	1
8	$W_2 = P_8$	$w_2 = 1$	2
10	$W_3 = P_{10}P_9$	$w_3 = 2$	3
11	$W_4 = P_{11}$	$w_4 = 1$	3

For $k = 12$, $W_{10} = P_{14}P_{13}P_{15}P_{16}P_{12}$ with $w_{10} = 3$ and end node 4.

For $k = 17$, $W_{12} = P_{19}P_{18}P_{20}P_{21}P_{17}$ with $w_{12} = 3$ and end node 4.

End node	Score
1	2
2	1
3	2
4	4

Applying (iii) in procedure POLYGON, we have the resulting graph in figure 12 with $k_1 = 4$.

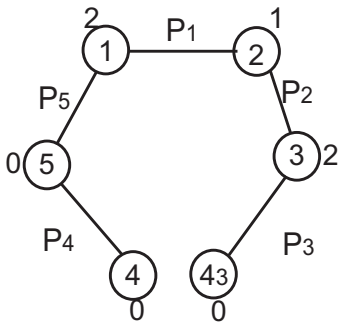


Figure 12: Tree after procedure POLYGON is applied

$L_1 = \{W_{10}, W_{12}\}$ and $L_2 = \{W_1, W_3, W_2, W_4\}$.

Step 1: Walks determined from nodes 4 and 4_3:

starting node	W_i	w_i	end node
4	$W_{13} = P_4P_5$	$w_{13} = 2$	1
4_3	$W_{14} = P_3$	$w_{14} = 1$	3

and $L_3 = \{W_{13}, W_{14}\}$.

Step 2: Delete nodes 4, 4_3, 5 and patterns $P_3P_4P_5$. The resulting graph is left with nodes 1, 2, 3 and patterns P_1P_2. L = {1, 3}.

End node	Score
1	3
2	1
3	2

Step 3: $k = 3$

3.1 Starting from node 3, $W_{15} = P_2$ with $w_{15} = 1$ and end node 2. S(2) = 1, affixing is not needed.

3.2 Using procedure CONSTRUCT2: Extended walk $W_{16} = W_3W_4\text{rev}W_{14} = P_{10}P_9P_{11}P_3$ with $w_{16} = 2$ and end node 2.

3.3 Walks deleted: $W_3W_4W_{14}$, $\mathrm{L}_2 = \{W_1, W_2, W_{15}\}$ and $\mathrm{L}_3 = \{W_{13}, W_{16}\}$. Return to step 2.

Step 2: Delete node 3 and pattern P_2. The resulting graph is left with nodes 1, 2 and pattern P_1. L = {1, 2}.

End node	Score
1	3
2	2

Step 3: $k = 2$

3.1 Starting from node 2, $W_{17} = P_1$ with $w_{17} = 1$ and end node 1. S(1) = 3, affixing is not needed.

3.2 Using procedure CONSTRUCT2: Extended walk W_{18} =$W_{16}W_{15}\mathrm{rev}W_2$ = $P_{10}P_9P_{11}P_3P_2P_8$ with $w_{18} = 2$ and end node 1.

3.3 Walks deleted: $W_2W_{15}W_{16}$, $\mathrm{L}_2 = \{W_1, W_{17}\}$ and $\mathrm{L}_3 = \{W_{13}, W_{18}\}$. Return to step 2.

Step 2: Delete node 2 and pattern P_1. The resulting graph is left with a single node 1.

End node	Score
1	3

Step 3: Single node $k = 1$ is left. $\mathrm{L}_1 = \{W_{10}, W_{12}\}$, $\mathrm{L}_2 = \{W_1, W_{17}\}$ and $\mathrm{L}_3 = \{W_{13}, W_{18}\}$.

$W_v = W_1$ with $w_1 = 2$, $W_s = W_{18}$ with $w_{18} = 2$ and $W_t = W_{13}$ with $w_{13} = 2$. Since $w_{10} = w_{12} > w_{13}$ and $w_{10} = w_{12} > w_1$, then applying procedure CONSTRUCT1, we have

a. The sequence of walks with end node $k_1 = 4$ is equal to $W_{10}\mathrm{rev}W_{12}$ = $P_{14}P_{13}P_{15}P_{16}P_{12}P_{17}P_{21}P_{20}P_{18}P_{19}$,

b. The sequence of walks with end node $k = 1$ is equal to $W_{18}W_{13}W_{17}\mathrm{rev}W_1$ = $P_{10}P_9P_{11}P_3P_2P_8P_4P_5P_1P_6P_7$ and

c. Then, the optimal sequence equals $W_{10}\text{rev}W_{12}W_{18}W_{13}W_{17}\text{rev}W_1 = P_{14}P_{13}P_{15}P_{16}P_{12}P_{17}P_{21}P_{20}P_{18}P_{19}P_{10}P_9P_{11}P_3P_2P_8P_4P_5P_1P_6P_7$ with a maximum number of open stacks equal to 4.

3 The heuristic of minimal cost node

This section presents the heuristic of minimal cost node that uses the idea of visiting each arc in G_p once. We begin with definitions of terms and introduce the heuristic.

Definition 5 *Arc contraction is an operation by removing an arc (u, v) from a graph G and joining nodes u and v into a single node. All other arcs incident to u or v become incident to the single node.*

Definition 6 *A graph G is isomorphic to a graph H if there exists a one-to-one function Ψ from $V(G)$, the vertex set of G, onto $V(H)$ such that arc $(u, v) \in E(G)$, the arc set of G, if and only if $(\Psi(u), \Psi(v)) \in E(H)$.*

Definition 7 *A graph G_p is called a **minor** of a graph G if G_p is isomorphic to a graph that can be obtained by zero or more arc contractions from G in any order.*

Consider the graphs G and G_p as illustrated in figure 13. The graph G_p is obtained from G by arc contraction of arcs (m, n) and (r, s). Thus, graph G_p is a minor of G.

The heuristic of minimal cost node uses arcs in G_p, a minor of graph G, as the basis for determining the least number of arcs to be sequenced in order to close the node. The order in which the arcs are removed from G_p follows the order in which the patterns are sequenced [12].

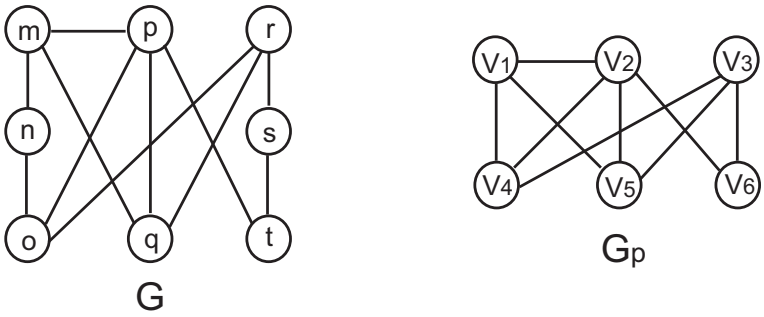


Figure 13: Illustration of a minor of G

Algorithm

To begin with the algorithm, we denote the following:

- V is a set of nodes of G_p;
- E is the set of arcs of G_p;
- $\Omega(k)$ is the degree of the node $k \in$ V excluding the arcs incident to k already visited;
- SETV is an ordered set of nodes of V yet to be sequenced. The nodes are ordered in an increasing order of degree $\Omega(k)$. A node $k \in V$ is in SETV if and only if $\Omega(k) \geq 1$;
- ARC is an ordered set of arcs of G_p already visited;
- OPEN is the set of nodes in V already open (at least one arc incident to this node is in set ARC and not all arcs incident to this node is in the set ARC);
- s is the number of arcs in ARC (the number of arcs already visited);
- ξ' is the maximum number of open stacks generated by the heuristic.

For a given graph G_p, we sequence the patterns to be cut in the following steps:

Step 0: Initialization. Label each node of G_p with its corresponding degree Ω. OPEN and ARC are empty. Set ξ' and s equal to zero.

Step 1: List in SETV all the nodes from G_p in non-decreasing order of degree Ω. Select two adjacent nodes n and m in G_p that have the smallest degree Ω. List these nodes in OPEN and arc (n, m) in ARC. Delete arc (n, m) in G_p and with the resulting graph, update the new degree Ω of nodes n and m. Set ξ' $= 2$ and $s = 1$.

Step 2: List in SETV all the nodes left in the graph. If SETV is empty, then STOP. Otherwise, find a node p with the smallest degree Ω adjacent to the nodes in OPEN. If there are more than one adjacent nodes, choose a node p adjacent to a node with the smallest degree Ω) in OPEN. Add node p in OPEN. Add the arcs incident to node p in ARC and delete these arcs in the graph. With the resulting graph, label each node with its corresponding degree Ω. If $\Omega(n)$ $= 0$, then delete this node in OPEN and in the graph.

Step 3: List the nodes in OPEN excluding the nodes with degree $\Omega = 0$ and ARC including the new arcs. Generate the new ξ' and s. Return to step 2.

To illustrate the heuristic with an example, consider the MOSP graph $G = G_p$ in figure 14 where each pattern has two piece types. $V = \{1, 2, ..., 9\}$ and $E = \{(1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (2,4), (2,6), (3,7), (3,9),$
$(4,6), (5,7), (5,8), (8,9), (6,9), (7,9), (8,9)\}$.

Step 0: Figure 15 shows the initialization. OPEN = {} and ARC = {}. ξ' $= 0$ and $s = 0$.

Step 1: SETV = {2, 3, 4, 8, 5, 6, 7, 9, 1}. Two adjacent nodes n = 2 and m = 4. OPEN = {2, 4} and ARC = {(2, 4)}. ξ' $= 2$ and $s = 1$. The resulting graph is in figure 16.

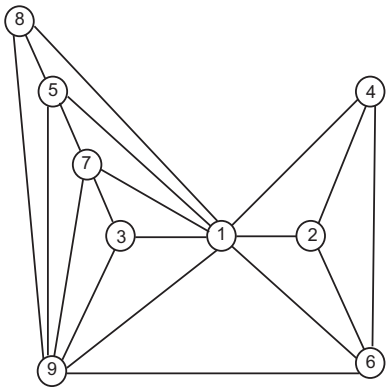


Figure 14: Graph G_p for the example

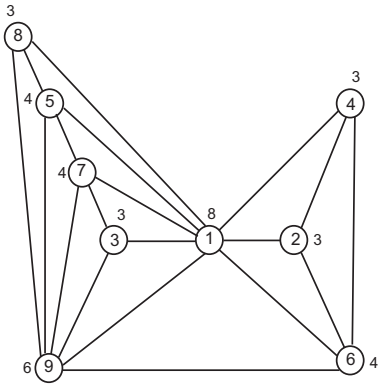


Figure 15: Initialization

Step 2: SETV = {2, 4, 3, 8, 5, 6, 7, 9, 1}. p = 6, add node 6 in OPEN. Add arcs (2, 6) and (4, 6) in ARC. Delete these arcs, the resulting graph is in figure 17.

Step 3: OPEN = {2, 4, 6} and ARC = {(2, 4), (2, 6), (4, 6)}. ξ' = 3 and s = 3. Return to step 2.

Step 2: SETV = {2, 4, 6, 3, 8, 5, 7, 9, 1}. p = 1, add node 1 in OPEN. Add arcs (1, 4), (1, 2), (1, 6) and delete these arcs in the graph. The resulting graph is in figure 18. The degree $\Omega(n = 2, 4) = 0$, delete nodes 2 and 4 in OPEN and in the graph.

Step 3: OPEN = {1, 6} and ARC = {(2, 4), (2, 6), (4, 6), 1, 4), (1, 2), (1, 6)}. ξ' = 4 and $s = 6$. Return to step 2.

Step 2: SETV = {6, 3, 8, 5, 7, 1, 9}. p = 9, add node 9 in OPEN. Add arcs (1,

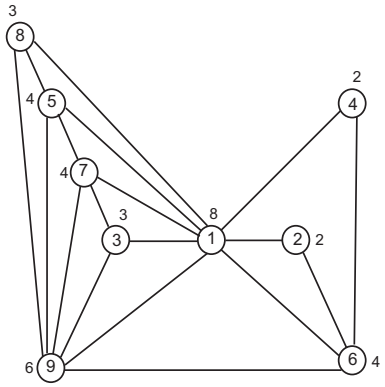


Figure 16: Resulting graph after arc (2,4) is deleted

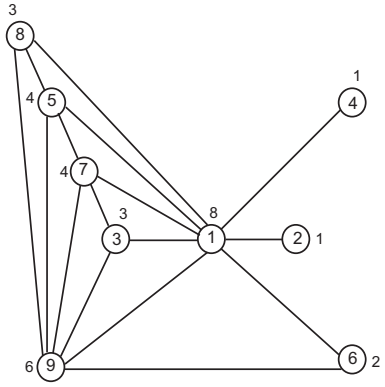


Figure 17: Resulting graph after arcs (2,4) and (2, 6) are deleted

9), (6, 9) and delete these arcs in the graph. The resulting graph is in figure 19.

Step 3: OPEN = {1, 9} and ARC = {(2, 4), (2, 6), (4, 6), 1, 4), (1, 2), (1, 6), (1, 9), (6, 9)}. ξ' = 3 and s = 8. Return to step 2.

Step 2: SETV = {8, 3, 1, 9, 5, 7}. p = 3, add node 3 in OPEN. Add arcs (1, 3), (3, 9) and delete these arcs in the graph. The resulting graph is left with nodes 1, 3, 5, 7, 8, and 9.

Step 3: OPEN = {1, 3, 9 } and ARC = {(2, 4), (2, 6), (4, 6), (1, 4), (1, 2), (1, 6), (1, 9), (6, 9), (1, 3), (3, 9) }. ξ' = 3 and s = 10. Return to step 2.

Step 2: SETV = {3, 1, 9, 8, 5, 7}. p = 7, add node 7 in OPEN. Add arcs (1, 7), (3, 7), (7, 9) and delete these arcs in the graph. The resulting graph is left with nodes 1, 5, 7, 8, and 9.

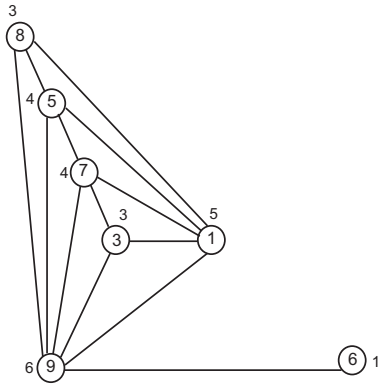


Figure 18: Resulting graph after arcs (1, 4), (1, 2), and (1, 6) are deleted

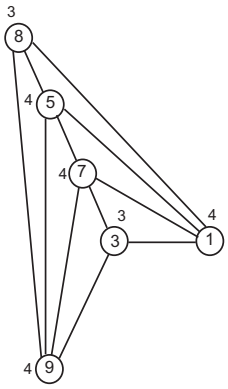


Figure 19: Resulting graph after arcs (1, 9) and (6, 9) are deleted

Step 3: OPEN = {1, 7, 9 } and ARC = {(2, 4), (2, 6), (4, 6), (1, 4), (1, 2), (1, 6), (1, 9), (6, 9), (1, 3), (3, 9), (1, 7), (3, 7), (7, 9) }. ξ' = 4 and s = 13. Return to step 2.

Step 2: SETV = {7, 1, 9, 8, 5}. p = 5, add node 5 in OPEN. Add arcs (1, 5), (5, 7), (5, 9) and delete these arcs in the graph. The resulting graph is left with nodes 1, 5, 8, and 9.

Step 3: OPEN = {1, 5, 9 } and ARC = {(2, 4), (2, 6), (4, 6), (1, 4), (1, 2), (1, 6), (1, 9), (6, 9), (1, 3), (3, 9), (1, 7), (3, 7), (7, 9), (1, 5), (5, 7), (5, 9)}. ξ' = 4 and s = 16. Return to step 2.

Step 2: SETV = {1, 5, 9, 8}. p = 8, add node 8 in OPEN. Add arcs (1, 8), (5, 8), (8, 9) and delete these arcs in the graph. No nodes left in the graph.

Step 3: OPEN = { } and ARC = {(2, 4), (2, 6), (4, 6), (1, 4), (1, 2), (1, 6), (1, 9), (6, 9), (1, 3), (3, 9), (1, 7), (3, 7), (7, 9), (1, 5), (5, 7), (5, 9), (1, 8), (5, 8), (8, 9)}. ξ' = 4 and $s = 19$. Return to step 2.

Step 2: SETV = { }. Stop.

Finally, we have an optimal sequence of arcs (2, 4), (2, 6), (4, 6), (1, 4), (1, 2), (1, 6), (1, 9), (6, 9), (1, 3), (3, 9), (1, 7), (3, 7), (7, 9), (1, 5), (5, 7), (5, 9), (1, 8), (5, 8), (8, 9) with a maximum number of open stacks equal to four.

4 The exact method

To have an idea of how we determine the exact method for solving a given MOSP, consider the graph G in figure 20. In this example, we can easily identify the maximum clique of size 4. Considering only this clique and deleting all other arcs and nodes of the original graph, the resulting graph has all nodes with degree 3. Hence, by proposition 3, we have at least 4 open stacks at some time during the cutting process. Moreover, by proposition 2, 4 is the lower bound on the number of open stacks for the original graph. The exact method follows this way of exploring the graph. However, for a more complicated graph, we need to explore the structure of graphs which holds "small" cliques as stated in section 2.

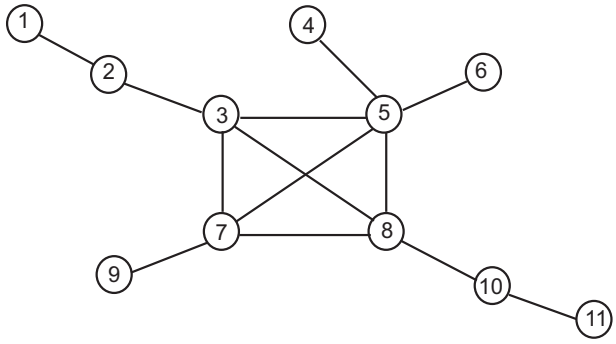


Figure 20: A graph to illustrate lower bound

In this section, we present an algorithm for solving the MOSP in a branch and

bound approach. It includes the concept of equivalent solutions, the algorithm for heuristic arc contraction that gives a guaranteed lower bound for a given minor of G and the process of the branch and bound scheme.

4.1 Equivalency proposition

The concept of the following proposition generates shortcuts in the enumeration for the optimal sequencing for a given G_p.

Let $N_b(i)$ be the "neighborhood" of a node i, the set of nodes that are adjacent to node i, and $\widetilde{N}_b(i) = N_b(i) \cup \{i\}$ in G_p. Then, we say that nodes i and j are equivalent whenever $N_b(i) = N_b(j)$ or $\widetilde{N}_b(i) = \widetilde{N}_b(j)$.

Equivalency proposition. *If G_p has two different nodes, say i and j, with $N_b(i) = N_b(j)$ or $\widetilde{N}_b(i) = \widetilde{N}_b(j)$, then from a given feasible solution of the problem where p_i is not immediately followed by p_j or vice versa, it is possible to construct a new sequence with p_i and p_j appearing consecutively. Moreover, the maximum number of open stacks by the new sequence is less than or equal to the original sequence.*

Proof. We assume without loss of generality that 1,..., i,..., j,..., n, $i \neq j$ is a feasible sequence of the n vertices of G_p, and $N_b(i) = N_b(j)$. When we sequence all the incident arcs to i, the set of open stacks associated to its neighboring vertices is the same set that would be opened by p_j. Therefore, it is possible to sequence j immediately before i, since the total number of open stacks remains the same. A similar argument can be used to prove the case when $\widetilde{N}_b(i) = \widetilde{N}_b(j)$. □

With equivalency proposition, we can reduce equivalent nodes in G_p to construct a new graph G_p in determining a new lower bound. Furthermore, it can assist in the enumeration of the feasible solution schema.

In figure 20, part (a), an initial G_p is given. Since nodes 2 and 4, nodes 5 and 3 are equivalent nodes, then we can reduce the initial G_p as shown in part (b). From

part (b), we see another equivalence among nodes 7 and 3. Then, we get another graph after reduction method is applied in part (c). Observe that $|G_p|$ decreases from 8 to 5, and the number of arcs decreases from 15 to 7. In this instance, a level of hierarchy on the equivalence relations exists that must be realized. Nodes 3 and 7 are equivalent only after the first reduction is applied. Hence, we can sequence 3 and 7 but node 3 must precede node 7 in the sequence.

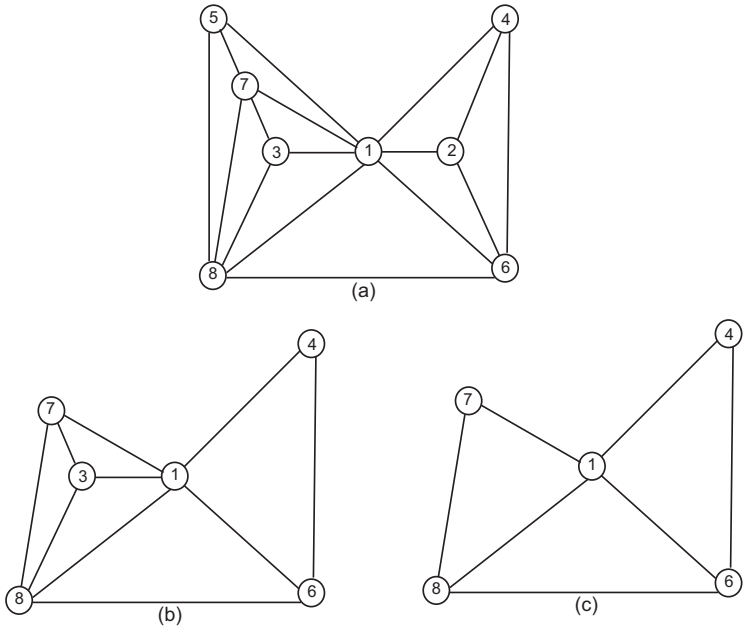


Figure 21: Identification of equivalencies

Therefore, if node i and j in G_p with the condition that $N_b(i) = N_b(j)$ or $\widetilde{N}_b(i) = \widetilde{N}_b(j)$, for the sake of finding an optimal sequencing, we can delete node i or node j and all arcs incident to it.

4.2 Heuristic arc contraction

We need to determine the best lower bound by following the algorithm for heuristic arc contraction. We denote the following:

- $lb_p = \max\{|P_j|,\ j = 1, 2, ..., n\}$ where $|P_j|$ = degree of the pattern P_j.

- $lb_d = 1 + \min \{\text{degree } (n) | \ n \text{ is a node of } G_p\}$,
- $lb_c = \max \{K_i \mid K_i \text{ is a minor of } G_p\}$,
- $|G|$ = order of G_p,
- $dmin$ = minimum degree of the nodes in G_p.

Bounds lb_p and lb_d are instant. Since for lb_p, the number of unfinished stacks can not be less than the maximum number of piece types of any pattern of a MOSP. lb_d is derived from the fact that a piece type p_i, of minimum degree in G_p, is incident to all those p_j's where p_i and p_j's belonging to the same pattern. Since some piece type has to be sequenced first at least this number of stacks will be opened.

The bound lb_c, as stated in [17], indicates that a lower bound is obtained if G_p is a complete graph.

Algorithm

Step 0 Initialization. Determine the lb_p, lb_d, $|G_p|$, and lb where $lb = \max \{lb_p, lb_d\}$.

Step 1 If G_p is a complete graph, then $lb_c = lb$. Stop. If $|G_p| \leq lb$, then $lb_c = lb$. Stop. Else, let S be the node set of G in non-decreasing order of degree (n).

Step 2 Choose an arc (i, j) in G_p that has the smallest node degree in S. Contract arc (i, j) and with the resulting graph, if $dmin \geq 2$ and $dmin + 1 > lb$, then the new $lb = dmin + 1$. Return to step 1.

Consider the following patterns: $P_1 = \{p_1, p_3, p_6\}$, $P_2 = \{p_1, p_4, p_5\}$, $P_3 = \{p_2, p_4, p_7\}$, $P_4 = \{p_2, p_5, p_6\}$, $P_5 = \{p_3, p_4\}$. The corresponding graph is shown in figure 22 where nodes 1, 2, 3, 4, 5, 6, 7 represent piece types $p_1, p_2, p_3, p_4, p_5, p_6, p_7$ respectively. The value for the lb_c is generated using the algorithm in the following way:

Step 0: Initialization. $lb_p = 3$, $lb_d = 3$, $|G| = 7$ and $lb = 3$.

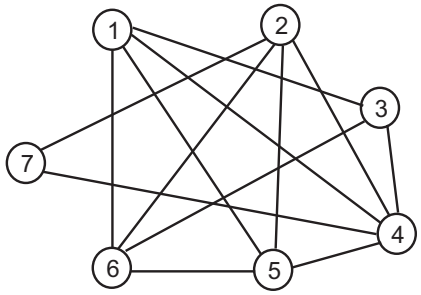


Figure 22: A graph to illustrate the Heuristic arc contraction algorithm

Step 1: G_p is not a complete graph. $|G_p| > lb$. $S = \{7, 3, 1, 2, 5, 6, 4\}$.

Step 2: Arc (7, 2) is selected. Contract arc (7, 2) and the resulting graph is shown in figure 23. The $dmin = 3$, thus the new $lb = 4$. Return to step 1.

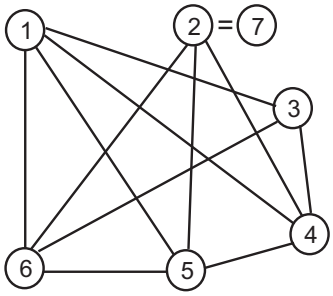


Figure 23: A graph after first contraction

Step 1: $|G_p| = 6 > lb$. $S = \{2, 3, 1, 5, 6, 4\}$.

Step 2: Arc (3, 4) is selected. Contract arc (3, 4) and the resulting graph is shown in figure 24. The $dmin = 3$, thus the new lb is still 4. Return to step 1.

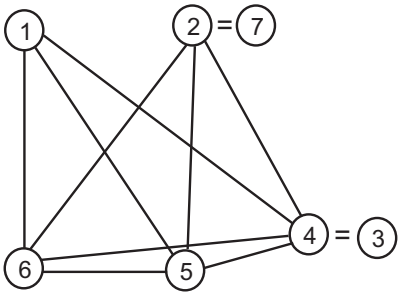


Figure 24: A graph after second contraction

Step 1: $|G_p| = 5 > lb$. $S = \{1, 2, 4, 5, 6\}$.

Step 2: Arc (1, 6) is selected. Contract arc (1, 6) and the resulting graph is shown in figure 25. The $dmin = 3$, thus the new lb is still 4. Return to step 1.

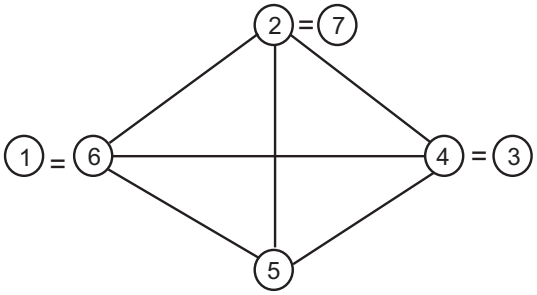


Figure 25: A graph after third contraction = K_4

Step 1: G_p is a complete graph. Thus, $lb_c = 4$. Stop.

The set S needs to be updated in every iteration to keep the lower bound as large as possible. For instance in a three-regular graph as shown in figure 26, if set S is sorted just once, two different cliques, K_3 and K_4 are obtained as illustrated in figures 27 and 28, respectively.

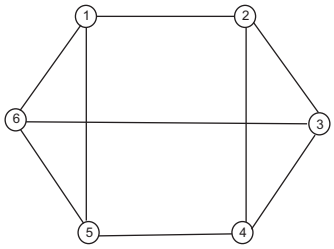


Figure 26: Initial graph G=G_p

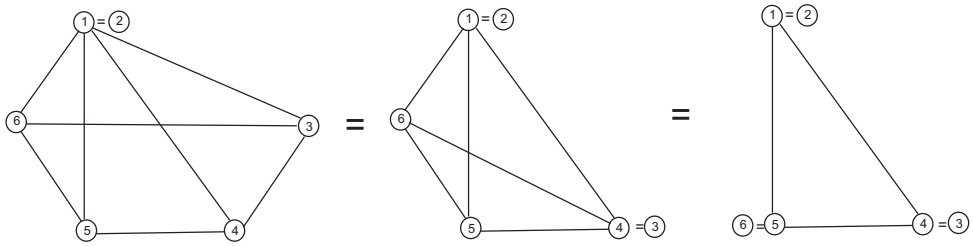


Figure 27: Minor of G generating K_3

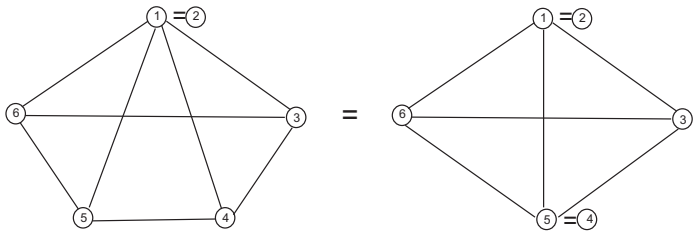


Figure 28: Minor of G generating K_4

4.3 A branch and bound scheme

We use a greedy method for the branch and bound scheme. By first branching from a node 0 and searching the next most favorable node for branching in order to find an optimal solution for a given MOSP. In branching, we need to choose a node that result in the least possible number of incomplete piece types. For illustration, we denote the following:

- S is the set of all piece types of the open stack problem;
- S_u^i is the set of unfinished piece types at branching node i, $i = 0, 1, 2, ..., n$;
- S_o^i is the set of currently open stacks at branching node i where $S_o \subseteq S_u \subseteq S$;
- LB_1 is the maximum number of open piece types in each pattern;
- v_j^k is the number of open stacks that results when going from node j to k, a successor of j;
- $LB_2 = min\{v_j^k\}$;
- LB_3 is the number of open stacks that resulted when branching from the predecessor of j. If p is the predecessor of j then the lower bound is v_j^k;
- $LB^j = max\{LB_2, LB_3, LB^p\}$ $j > 0$, where p is the predecessor of node j, and the overall lower bound for node j including its predecessor node;

- $LB^0 = max\{LB_1, LB_2\}$;
- $UP^j = |S_u^i|$ = upper bound on the number of open stacks at each node which is the total number of piece types of the problem still left to be cut.
- v^{up} is the value of the best solution so far for the open stacks problem. Initially, $v^{up} = |S|$.

In the branching scheme, each node i will be identified by the sets S_u^i and S_o^i (see figure 29).

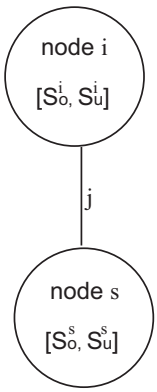


Figure 29: Identification of a node in the branching scheme

For each piece type $p_j \in S_u^i$, create a new branching node say s, with $S_u^s = S_u^i \setminus \{p_j\}$ and S_o^s is the set of open unfinished stacks that is obtained from all the patterns containing piece type p_j that are sequenced.

We start the enumeration at node 0 with starting sets $S_o^0 = \varnothing$ and $S_u^0 = S = \{1, 2, ..., m\}$ (see figure 30). Then choose the most favorable node using a greedy criterion for branching. When any lower bound in node j is greater than or equal to v^{up}, node j can be fathomed. When $LB^j \geq UP^j$, node j can be fathomed and v^{up} is updated with LB^j if $v^{up} > LB^j$.

Consider for the following instance of a MOSP: $P_1 = \{p_1, p_2\}$, $P_2 = \{p_1, p_4\}$, $P_3 = \{p_1, p_3\}$, $P_4 = \{p_2, p_5\}$, $P_5 = \{p_3, p_4\}$, $P_6 = \{p_2, p_4\}$, $P_7 = \{p_3, p_5\}$, $P_8 = \{p_2, p_3\}$, $P_9 = \{p_1, p_5\}$, $P_{10} = \{p_2, p_6\}$. We start with node 0, where $S_o^0 = \emptyset$ and

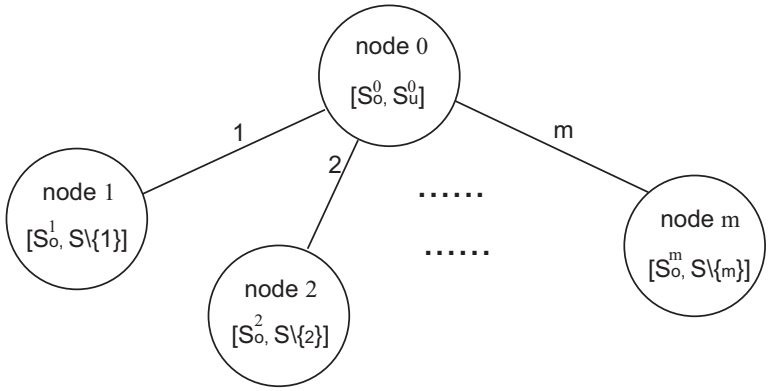


Figure 30: Branching from node 0

$S_u^0 = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. At node 0, $LB_1 = 2$, $UP^0 = 6$ and $v^{up} = 6$. Branching at node 0 produces 6 new nodes as shown in figure 31.

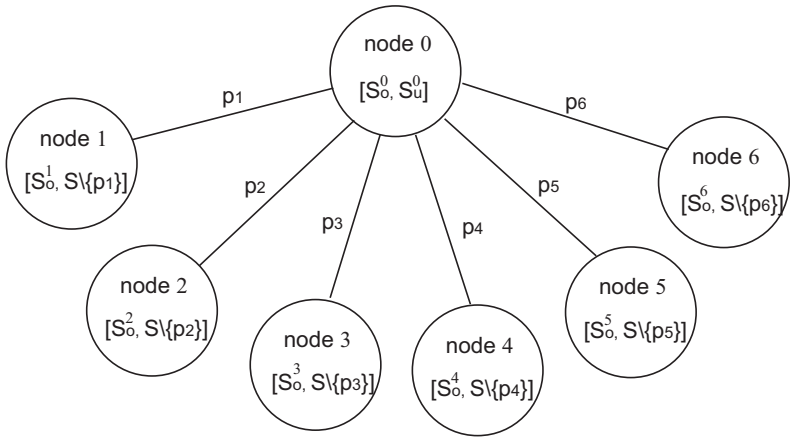


Figure 31: Branching from node 0

For these nodes we have:

$S_o^1 = \{p_2, p_3, p_4, p_5\}$, $UP^1 = 5$, $v_o^1 = 5$, $LB_3 = 5$;

$S_o^2 = \{p_1, p_3, p_4, p_5, p_6\}$, $UP^2 = 5$, $v_o^2 = 6$, $LB_3 = 6$;

$S_o^3 = \{p_1, p_2, p_4, p_5\}$, $UP^3 = 5$, $v_o^3 = 5$, $LB_3 = 5$;

$S_o^4 = \{p_1, p_2, p_3\}$, $UP^4 = 5$, $v_o^4 = 4$, $LB_3 = 4$;

$S_o^5 = \{p_1, p_2, p_3\}$, $UP^5 = 5$, $v_o^5 = 4$, $LB_3 = 4$;

$S_o^6 = \{p_2\}$, $UP^6 = 5$, $v_o^6 = 2$, $LB_3 = 2$.

Hence, at node 0, $LB_2 = min\{v_o^i, i = 1, 2, ..., 6\} = 2$ and $LB^0 = max\{LB_1, LB_2\} =$

2.

Following the scheme we fathom nodes 1, 2, and 3 since the lower bound on these nodes equal the currently upper bound for the optimal value of the problem.

We then branch the most favorable node which is node 6(see figure 32).

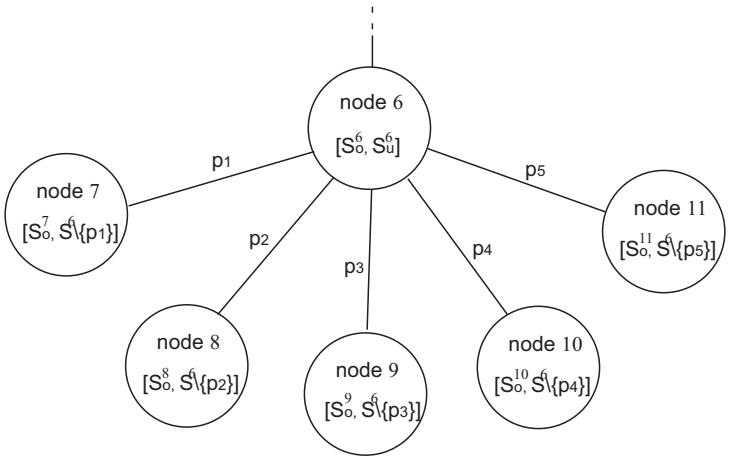


Figure 32: Branching from node 6

For these nodes we have:

$S_o^7 = \{p_2, p_3, p_4, p_5\}$, $v_o^7 = 5$;

$S_o^8 = \{p_1, p_3, p_4, p_5\}$, $v_o^8 = 5$;

$S_o^9 = \{p_1, p_2, p_4, p_5\}$, $v_o^9 = 5$;

$S_o^{10} = \{p_1, p_2, p_3\}$, $v_o^{10} = 4$;

$S_o^{11} = \{p_1, p_2, p_3\}$, $v_o^{11} = 4$.

Hence, at node 0, $LB_2 = min\{v_o^i, i = 1, 2, ..., 6\} = 4$ and $LB^6 = max\{LB_2, LB_3, LB^0\}$ $= 4$ and $UP^6 = 5$. Since $UP^6 > LB^6$, then we can not fathomed node 6 but nodes 7, 8 and 9 are fathomed.

We then branch the next favorable node which is node 10 and for these nodes we have:

$S_o^{12} = \{p_2, p_3, p_5\}$, $v_o^{12} = 4$;

$S_o^{13} = \{p_1, p_3, p_5\}$, $v_o^{13} = 4$;

$S_o^{14} = \{p_1, p_2, p_5\}$, $v_o^{14} = 5$;

$S_o^{15} = \{p_1, p_2, p_3\}$, $v_o^{15} = 5$.

Hence, at node 10, $LB_2 = 4$, $LB^{10} = 4$ and $UP^{10} = 4$. Since $LB^{10} = UP^{10}$, any solution from this node and through has an optimal value of 4. We, therefore, update v^{up} to 4 and fathom node 10 and consequently, all its successors. Nodes 4 and 5 are also fathomed since their lower bounds equal to the current upper bound. Then the procedure ends with an optimal value of 4 and the optimal sequence follows by sorting the patterns that contain piece type p_6 first, the patterns that contain piece type p_5 is next and the patterns that complete any other uncompleted piece types until all piece types are completed. One possible solutions for the sequence is $P_{10}P_4P_7P_9P_1P_2P_3P_5P_6P_8$ with a lower bound 4.

4.4 Exact Method: branch and bound approach

In practice, the following algorithm can take a long time to solve larger number of patterns that forms a large and overlapping cliques.

Algorithm

Step 0 Determine the graph G. All components that are trees are removed and call this removed set D.

Step 1 Execute the pre-processing on G to obtain G_p following section 2.

Step 2 Execute the Heuristic of Minimal Cost Node to obtain ξ' and the sequence of patterns following section 3.

Step 3 Execute the Heuristic Arc Contraction to get the lower bound lb_c following subsection 4.2.

Step 4 If $\xi' = lb_c$ (upper bound = lower bound) then the optimal solution is found in step 2. Else, look for the equivalence set E and implement the reduction process among the equivalent nodes as illustrated in subsection 4.1 to construct

a new graph G_p. Then execute the Heuristic Arc Contraction to get the new lb_c. After which, execute a branch and bound in order to obtain the optimal solution and insert the deleted equivalent nodes.

Step 5 Insert the vertices of D into the final sequence.

To illustrate the algorithm, consider the following instance of MOSP:

$P_1 = \{p_1, p_3\}$, $P_2 = \{p_2, p_3\}$,$P_3 = \{p_3, p_5\}$, $P_4 = \{p_4, p_5\}$, $P_5 = \{p_5, p_6\}$,
$P_6 = \{p_5, p_7\}$, $P_7 = \{p_5, p_{10}\}$, $P_8 = \{p_5, p_{12}\}$, $P_9 = \{p_5, p_{11}\}$, $P_{10} = \{p_6, p_9\}$,
$P_{11} = \{p_6, p_{11}\}$, $P_{12} = \{p_6, p_8\}$, $P_{13} = \{p_6, p_{10}\}$, $P_{14} = \{p_6, p_7\}$, $P_{15} = \{p_7, p_8\}$,
$P_{16} = \{p_7, p_{12}\}$, $P_{17} = \{p_7, p_{10}\}$, $P_{18} = \{p_8, p_9\}$, $P_{19} = \{p_9, p_{12}\}$, $P_{20} = \{p_9, p_{11}\}$,
$P_{21} = \{p_8, p_{11}\}$, $P_{22} = \{p_{10}, p_{11}\}$, $P_{23} = \{p_{11}, p_{12}\}$, $P_{24} = \{p_8, p_{12}\}$, $P_{25} = \{p_8, p_{10}\}$,
$P_{26} = \{p_{10}, p_{13}\}$, $P_{27} = \{p_{13}, p_{14}\}$, $P_{28} = \{p_{13}, p_{15}\}$, $P_{29} = \{p_{13}, p_{16}\}$.

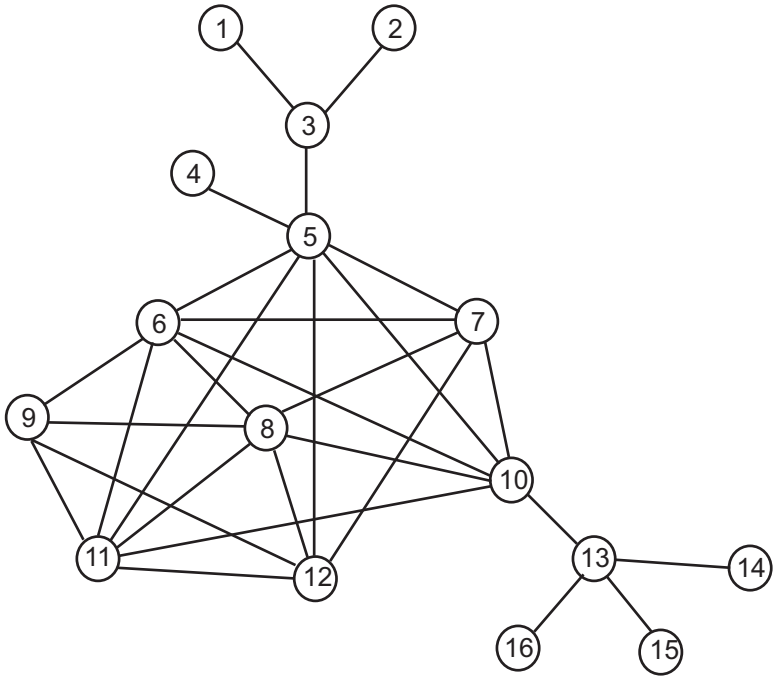


Figure 33: Graph that illustrates the branch and bound

Step 0: The graph is shown in figure 30 where p_i represents node i. Removed set $D = \{P_1, P_2, P_3, P_4, P_{26}, P_{27}, P_{28}, P_{29}\}$.

Step 1: Pre-processing. No redundancy occurs among the patterns. The graph G shows two components as trees and when these trees are removed, we have G_p as shown in figure 31. We have optimal sequences $P_1P_2P_3P_4$ and $P_{26}P_{27}P_{28}P_{29}$ for the two trees that are removed which will be inserted in the final sequence with the same maximum number of open stacks equal to 2.

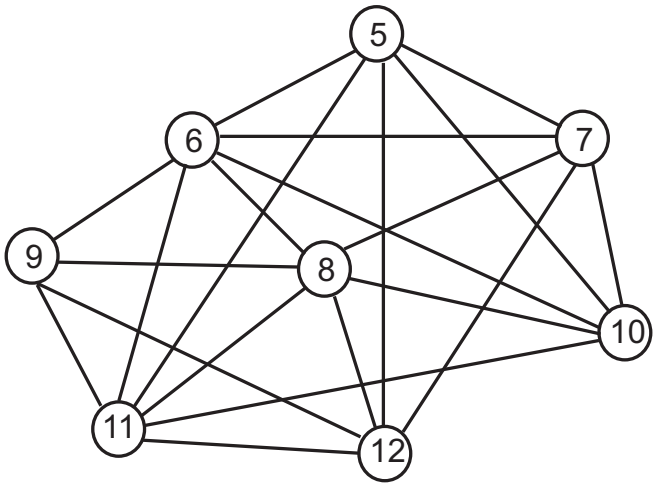


Figure 34: Graph G_p

Step 2: Heuristic of Minimal Cost Node generates $\xi' = 6$, $s = 21$ and one optimal sequence $P_{19}P_{20}P_{23}P_{18}P_{21}P_{24}P_{10}P_{11}P_{12}P_9P_5P_8P_{22}P_{25}P_{13}P_7P_{14}P_6P_{15}P_{16}P_{17}$.

Step 3: Heuristic Arc Contraction generates $lb_c = 6$.

Step 4: Since $\xi' = lb_c = 6$, then we have an optimal solution in step 2.

Step 5: Insert the patterns from D, we have a final sequence $P_{19}P_{20}P_{23}P_{18}P_{21}P_{24}P_{10}$ $P_{11}P_{12}P_9P_5P_8P_{22}P_{25}P_{13}P_7P_{14}P_6P_{15}P_{16}P_{17}P_{26}, P_{27}, P_{28}, P_{29}P_1P_2P_3P_4$ with a maximum number of open stacks 6.

5 Conclusion and future work

In this paper, the algorithms presented focus on the graphical model of a given MOSP. Each algorithm generates an optimal solution except for the heuristic of minimal cost node which generates the optimal sequence and its upper bound for

the maximum number of open stacks.

In some industrial environments such as glass, wood or steel industries, the optimal solutions using the heuristic are acceptable. The heuristic is a way to motivate a more efficient exact method for the MOSP with the utilization of a more profound computer programs, even though this is NP-hard problem [16].

In some real industrial settings, the maximum number of open stacks might be limited. And with this limitation, the optimal sequencing of the patterns may not satisfy the requirements. This arises to a more intricate study of two NP-hard problems, the cutting stock problem and the pattern sequencing problem. In [18], the formulation for the cutting and sequencing problem is suggested and in [19], the heuristic method for this problem is proposed.

REFERENCES

[1] Yanasse HH. Minimization of open orders-polynomial algorithms for some special cases. Pesquisa Operacional 1996;16(1):1-26.

[2] Dyson RG, Gregory AS. The cutting stock problem in the Kat glass industry. Operations Research Quarterly 1974;25:41-54.

[3] Madsen OBG. Glass cutting in a small ¡rm. Mathematical Programming 1979;17:85-90.

[4] Madsen OBG. An application of travelling-salesman routines to solve pattern-allocation problems in the glass industry. Journal of Operational Research Society 1988;39(3):249-56.

[5] Lins S. Traversing trees and scheduling tasks for duplex corrugator machines. Pesquisa Operacional 1989;9:40-54.

[6] Fink A, VoW S. Applications of modern heuristic search methods to pattern sequencing problems. Computer and Operations Research 1999;26(1):17-34.

[7] Foerster HE, WXascher G. Simulatedannealing for order spreadminimization in sequencing cutting patterns. European Journal of Operational Research 1998;110:272-81.

[8] Yuen BJ. Heuristics for sequencing cutting patterns. European Journal of Operational Research 1991;55:183-90.

[9] Yuen BJ. Improvedheuristics for sequencing cutting patterns. European Journal of Operational Research 1995;87: 57-64.

[10] Faggioli E, Bentivoglio CA. Heuristic andexact methods for the cutting sequencing problem. European Journal of Operational Research 1998;110(3):564-75.

[11] Yuen BJ, Richardson KV. Establishing the optimality of sequencing heuristics for cutting stock problems. European Journal of Operational Research 1995;84:590-8.

[12] Yanasse HH. A transformation for solving a pattern sequencing problem in the wood cut industry. Pesquisa Operacional 1997;17(1):57-70.

[13] Limeira MS. Development of an exact algorithm for solving a cutting pattern sequencing problem. Master, Brazilian Space Research Institute, Sao Jos)e dos Campos, Brazil, 1998 [in portuguese].

[14] Yanasse HH. On a pattern sequencing problem to minimize the maximum number of open stacks. European Journal of Operational Research 1997;100:454-63.

[15] Becceneri JC. The pattern sequencing problem to minimize the maximum number of open stacks in industrial cutting environments. Dissertation, Aeronautic Institute of Technology, Brazil, 1999 [in portuguese].

[16] Linhares A, Yanasse HH. Connections between cutting-pattern sequencing, VLSI design, and Kexible machines. Computers andOperations Research 2002;29(12):1759-72.

[17] Yanasse HH, Becceneri JC, Soma NY. Bounds for a problem of sequencing patterns. Pesquisa Operacional 1999;19(2):249-77.

[18] Yanasse HH, Pinto MJ. A cutting andsequencing integratedmod el. The Sixteenth Triennial Conference of the International Federation of Operational Research Societies, Edinburgh, Scotland, 2002. Programme, p. 121.

[19] Pileggi GCF. Approaches for the optimization of the cutting patterns generation andsequencing integratedproblems. Dissertation, University of Sao Paulo, Brazil, 2002 [in portuguese].